Universality of AdaGrad Stepsizes for Stochastic Optimization: Inexact Oracle, Acceleration and Variance Reduction

Anton Rodomanov (CISPA)

Joint work with Xiaowen Jiang (CISPA) and Sebastian Stich (CISPA)

20 June 2024
FGS Conference on Optimization
Gijón, Spain

Motivation

Stochastic Convex Optimization

Problem:

$$f^* = \min_{x \in Q} f(x),$$

where $f : \mathbb{R}^d \to \mathbb{R}$ is a convex function, $Q \subseteq \mathbb{R}^d$ is a simple convex set.

Stochastic gradient oracle: Random vector $g(x, \xi) \in \mathbb{R}^d$ (ξ is a r.v.),

$$\mathbb{E}_\xi[g(x, \xi)] = \nabla f(x).$$

Main example: $f(x) = \mathbb{E}_\xi[f(x, \xi)]$. Then, $g(x, \xi) = \nabla_x f(x, \xi)$.

E.g.: $f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x) \implies g(x, \xi) = \frac{1}{b} \sum_{j=1}^{b} \nabla f_{\xi_j}(x)$, where $\xi = (\xi_1, \ldots, \xi_b)$ with i.i.d. components from $\mathrm{Unif}(\{1, \ldots, n\})$.

Stochastic Gradient Method (SGD)

Problem: $f^* = \min_{x \in Q} f(x)$.

Stochastic Gradient Method (SGD):

$$x_{k+1} = \pi_Q(x_k - h_k g_k), \quad g_k \cong \hat{g}(x_k),$$

where $\pi_Q(x) = \operatorname{argmin}_{y \in Q} \|x - y\|$ is the Euclidean projection onto Q.

Main questions:

- How to choose step sizes h_k?
- What is the rate of convergence?

Convergence Guarantees for SGD

Assume that:

- Q is bounded: $\|x - y\| \le D, \ \forall x, y \in Q$.
- Variance of $\hat{g}$ is bounded: $\mathbb{E}_\xi[\|g(x,\xi) - \nabla f(x)\|_*^2] \le \sigma^2, \ \forall x \in Q$.

Nonsmooth optimization: $\|\nabla f(x)\|_* \le L_0$, $\forall x \in Q$.

$$h_k = \frac{D}{\sqrt{(L_0^2 + \sigma^2)(k+1)}} \quad \implies \quad \mathbb{E}[f(\bar{x}_k)] - f^* \le O\Big(\frac{(L_0 + \sigma)D}{\sqrt{k}}\Big),$$

where $\bar{x}_k = \frac{1}{k}\sum_{i=0}^{k-1} x_i$.

Smooth optimization: $\|\nabla f(x) - \nabla f(y)\|_* \le L_1\|x - y\|$, $\forall x, y \in Q$.

$$h_k = \min\Big\{\frac{1}{2L_1}, \frac{D}{\sigma\sqrt{k+1}}\Big\} \quad \implies \quad \mathbb{E}[f(\bar{x}_k)] - f^* \le O\Big(\frac{L_1 D^2}{k} + \frac{\sigma D}{\sqrt{k}}\Big).$$

Discussion

- What we saw previously is the standard approach in Optimization:
  1. Fix a certain Problem class $\mathcal{P}$.
  2. Develop a "good" method tailored to $\mathcal{P}$.
- However:
  - A specific problem may belong to multiple problem classes.
  - Different problems may belong to different problem classes.
- Ideally, we would like to have universal algorithms suitable for multiple problem classes at the same time.

Universal Gradient Methods [Nesterov 2015]

Problem: $\min_{x \in Q} f(x)$.

Hölder constants: $H_\nu := \sup\limits_{x,y \in Q; x \neq y} \frac{\|\nabla f(x) - \nabla f(y)\|_*}{\|x-y\|^\nu}, \ \nu \in [0,1]$.

Note:

- $\nu = 1$: $\|\nabla f(x) - \nabla f(y)\|_* \leq H_1 \|x - y\|$ (Lipschitz gradient).
- $\nu = 0$: $\|\nabla f(x) - \nabla f(y)\|_* \leq H_0$ (contains Lipschitz functions). This class is better than $\|\nabla f(x)\|_* \leq L_0$.
- If $H_{\nu_1}, H_{\nu_2} < +\infty$ for some $\nu_1 \leq \nu_2$, then $H_\nu < +\infty, \forall \nu \in [\nu_1, \nu_2]$.

Main assumption: There exists $\nu \in [0,1]$ such that $H_\nu < +\infty$.

Universal Gradient Methods – II

Method: $x_{k+1} = \pi_Q(x_k - \frac{1}{M_k}\nabla f(x_k))$, where M_k is found by line search to satisfy the following condition:

$$f(x_{k+1}) \leq f(x_k) + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{M_k}{2}\|x_{k+1} - x_k\|^2 + \frac{\epsilon}{2}.$$

Efficiency bound: $O\left(\inf_{\nu \in [0,1]} \left(\frac{H_\nu}{\epsilon}\right)^{\frac{2}{1+\nu}} D^2\right)$ iters to $f(x_k^*) - f^* \leq \epsilon$.

Universal Fast Gradient Method: $O\left(\inf_{\nu \in [0,1]} \left(\frac{H_\nu D^{1+\nu}}{\epsilon}\right)^{\frac{2}{1+3\nu}}\right)$.

Great methods but they do not work with stochastic oracle!

AdaGrad Methods

AdaGrad [McMahan and Streeter 2010; Duchi et al. 2011]: $(g_k \cong \hat{g}(x_k))$

$$x_{k+1} = \pi_Q(x_k - h_k g_k), \qquad h_k = \frac{D}{\sqrt{\sum_{i=0}^{k} \|g_i\|_*^2}}.$$

Foundation of nowadays popular Adam, RMSProp, ...

Convergence rate [Levy et al. 2018]: If $\nabla f(x^*) = 0$, then

$$\mathbb{E}[f(\bar{x}_k)] - f^* \leq O\left(\min\left\{\frac{L_0 D}{\sqrt{k}}, \frac{L_1 D^2}{k}\right\} + \frac{\sigma D}{\sqrt{k}}\right),$$

(L_0, L_1 are the Lipschitz constants of f, ∇f; σ is the variance.)

UniXGrad [Kavis et al. 2019]: Accelerated gradient method with AdaGrad step sizes based on difference of gradients. Convergence rate:

$$O\left(\min\left\{\frac{L_0 D}{\sqrt{k}}, \frac{L_1 D^2}{k^2}\right\} + \frac{\sigma D}{\sqrt{k}}\right).$$

Motivation and Related Work

Develop "fully universal" gradient methods that automatically adjust to the right Hölder class and oracle's variance.

Related work:

- Universal methods with line search [Nesterov 2015; Grapiglia and Nesterov 2017; Grapiglia and Nesterov 2020; Doikov and Nesterov 2021; Doikov, Mishchenko, et al. 2024]. Only for deterministic optimization.
- Adaptive methods for stochastic optimization [McMahan and Streeter 2010; Duchi et al. 2011; Levy et al. 2018; Kavis et al. 2019; Ene et al. 2021] No specific guarantees for Hölder class.
- Parameter-free methods [Orabona 2014; Cutkosky and Boahen 2017; Cutkosky and Orabona 2018; Jacobsen and Cutkosky 2023; Carmon and Hinder 2022; Defazio and Mishchenko 2023] Slightly different focus, also no specific guarantees for Hölder class (with stochastic oracle).

Universal Stochastic Gradient Methods [Rodomanov et al. 2024]

Problem: $\min_{x \in \operatorname{dom}\psi} [F(x) = f(x) + \psi(x)]$, f and ψ are convex, ψ is simple.

Assumptions:

1. Hölder gradient: $\|\nabla f(x) - \nabla f(y)\|_* \le H_\nu \|x - y\|^\nu$, $\nu \in [0, 1]$.
2. Bounded domain: $\|x - y\| \le D, \quad \forall x, y \in \operatorname{dom}\psi$.
3. Stochastic oracle: $\mathbb{E}_\xi[g(x, \xi)] = \nabla f(x)$, $\mathbb{E}_\xi[\|g(x, \xi) - \nabla f(x)\|_*^2] \le \sigma^2$.

Methods using (modified) AdaGrad stepsizes and needing to know only D:

- Basic method: $O\Big(\inf_{\nu \in [0,1]} \frac{H_\nu D^{1+\nu}}{k^{(1+\nu)/2}} + \frac{\sigma D}{\sqrt{k}}\Big)$.
- Accelerated method: $O\Big(\inf_{\nu \in [0,1]} \frac{H_\nu D^{1+\nu}}{k^{(1+3\nu)/2}} + \frac{\sigma D}{\sqrt{k}}\Big)$.

This work: Show that AdaGrad stepsizes are even more universal.

Main Algorithms and Results for Uniformly Bounded Variance

Problem Formulation – I: Approximate Smoothness

Problem: $\min_{x \in \operatorname{dom}\psi} [F(x) = f(x) + \psi(x)]$, f and ψ are convex, ψ is simple.

Main assumption: f is approximately smooth: there exist $L_f, \delta_f \geq 0$ and $\bar{f} \colon \mathbb{R}^d \to \mathbb{R}$, $\bar{g} \colon \mathbb{R}^d \to \mathbb{R}^d$ such that, for any $x, y \in \mathbb{R}^d$, we have

$$0 \leq \left[\beta_{f,\bar{f},\bar{g}}(x, y) := f(y) - \bar{f}(x) - \langle \bar{g}(x), y - x \rangle\right] \leq \frac{L_f}{2}\|y - x\|^2 + \delta_f.$$

NB: This is the (δ, L)-oracle introduced by [Devolder et al. 2013].

Examples:

- f is L-smooth $\iff$ $(\bar{f}, \bar{g}) = (f, \nabla f)$ with $L_f = L$, $\delta_f = 0$
- f is (ν, H_ν)-Hölder smooth $\implies$ $(\bar{f}, \bar{g}) = (f, \nabla f)$ with $L_f = [\frac{1-\nu}{2(1+\nu)\delta_f}]^{\frac{1-\nu}{1+\nu}} H_\nu^{\frac{2}{1+\nu}}$ and any $\delta_f > 0$.
- $\phi(x) \leq f(x) \leq \phi(x) + \delta$, $\forall x$, with L-smooth ϕ $\implies$ $(\bar{f}, \bar{g}) = (\phi, \nabla\phi)$ with $L_f = L$, $\delta_f = \delta$.
- $f(x) = \max_u \Psi(x, u)$ with str. concave Ψ, $\bar{u}(x) \approx_\delta \operatorname{argmax}_u \Psi(x, u)$ $\implies$ $\bar{f}(x) = \Psi(x, \bar{u}(x))$, $\bar{g}(x) = \nabla_u \Psi(x, \bar{u}(x))$ with $\delta_f = f$.

Problem Formulation – II

Problem: $F^* = \min_{x \in \operatorname{dom} \psi} [F(x) = f(x) + \psi(x)]$.

Assumptions:

1. f is (δ_f, L_f)-approximately smooth with components $(\bar{f}, \bar{g})$.
2. f can be accessed only via unbiased stochastic oracle $\hat{g}$ for $\bar{g}$: $\mathbb{E}_\xi[g(x, \xi)] = \bar{g}(x)$.
3. Uniformly bounded variance: $\operatorname{Var}_{\hat{g}}(x) := \mathbb{E}_\xi[\|g(x,\xi) - \bar{g}(x)\|_*^2] \le \sigma^2$.
4. Bounded domain: $\|x - y\| \le D$, $\forall x, y \in \operatorname{dom} \psi$.

Note: Asm. 4 can always be ensured with $D = 2R_0$ whenever we know $R_0 \ge \|x_0 - x^*\|$ by considering $F^* = \min_{x \in \operatorname{dom} \psi_D} [F_D(x) = f(x) + \psi_D(x)]$, where $\psi_D = \psi + \operatorname{Ind}_{B_0}$ with $B_0 = \{x : \|x - x_0\| \le R_0\}$.

Basic Universal Gradient Method

Algorithm 1 $\mathrm{UniSgd}_{\hat{g},\psi}(x_0; D)$

$g_0 \cong \hat{g}(x_0)$.
for $k = 0, 1 \ldots$ **do**
$\quad x_{k+1} = \mathrm{Prox}_\psi(x_k, g_k, M_k), \quad g_{k+1} \cong \hat{g}(x_{k+1})$.
$\quad M_{k+1} = \sqrt{M_k^2 + \frac{1}{D^2}\|g_{k+1} - g_k\|_*^2}$.

Prox-mapping: $\mathrm{Prox}_\psi(x, g, M) = \operatorname*{argmin}_{y \in \operatorname{dom}\psi} \{\langle g, y\rangle + \psi(y) + \frac{M}{2}\|y - x\|^2\}$.

Output point: $\bar{x}_k = \frac{1}{k}\sum_{i=1}^{k} x_i$.

Convergence rate: $\mathbb{E}[F(\bar{x}_k)] - F^* \leq O\left(\frac{L_f D^2}{k} + \frac{\sigma D}{\sqrt{k}} + \delta_f\right)$.

Accelerated Universal Gradient Method

Algorithm 2 $\mathrm{UniFastSgd}_{\hat{g},\psi}(x_0; D)$

$$v_0 = x_0,\ M_0 = A_0 = 0.$$

for $k = 0, 1, \ldots$ **do**

$$a_{k+1} = \tfrac{1}{2}(k+1),\ A_{k+1} = A_k + a_{k+1}$$

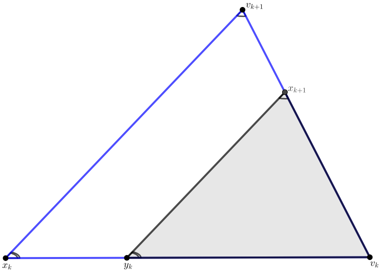
$$y_k = \tfrac{A_k}{A_{k+1}} x_k + \tfrac{a_{k+1}}{A_{k+1}} v_k,\quad g_{y_k} \cong \hat{g}(y_k).$$

$$v_{k+1} = \mathrm{Prox}_\psi(v_k, g_{y_k}, M_k / a_{k+1}).$$

$$x_{k+1} = \tfrac{A_k}{A_{k+1}} x_k + \tfrac{a_{k+1}}{A_{k+1}} v_{k+1}.$$

$$g_{x_{k+1}} \cong \hat{g}(x_{k+1}).$$

$$M_{k+1} = \sqrt{M_k^2 + \tfrac{a_{k+1}^2}{D^2} \| g_{x_{k+1}} - g_{y_k} \|_*^2}.$$



Convergence rate: $\mathbb{E}[F(x_k)] - F^* \leq O\left(\frac{L_f D^2}{k^2} + \frac{\sigma D}{\sqrt{k}} + k\delta_f\right).$

Example: Hölder Smooth Functions

Suppose f is (ν, H_ν)-Hölder smooth. Then, f as approximately smooth with $(\bar{f}, \bar{g}) = (f, \nabla f)$, arbitrary $\delta_f > 0$ and $L_f \sim [\frac{1}{\delta_f}]^{\frac{1-\nu}{1+\nu}} H_\nu^{\frac{2}{1+\nu}}$.

For UniSgd, we get, for $F_k = \mathbb{E}[F(\bar{x}_k)] - F^*$,

$$F_k \lesssim \frac{L_f D^2}{k} + \frac{\sigma D}{\sqrt{k}} + \delta_f \sim \frac{H_\nu^{\frac{2}{1+\nu}} D^2}{k \delta_f^{\frac{1-\nu}{1+\nu}}} + \frac{\sigma D}{\sqrt{k}} + \delta_f.$$

Minimizing this expression in δ_f, we get

$$F_k \leq O\left(\frac{H_\nu D^{1+\nu}}{k^{\frac{1+\nu}{2}}} + \frac{\sigma D}{\sqrt{k}}\right) \leq \epsilon \quad \text{in} \quad O\left(\left[\frac{H_\nu D^{1+\nu}}{\epsilon}\right]^{\frac{2}{1+\nu}} + \frac{\sigma^2 D^2}{\epsilon^2}\right) \text{ orac. calls.}$$

Similar reasoning for UniFastSgd gives, for $F_k = \mathbb{E}[F(x_k)] - F^*$,

$$F_k \leq O\left(\frac{H_\nu D^{1+\nu}}{k^{\frac{1+3\nu}{2}}} + \frac{\sigma D}{\sqrt{k}}\right) \leq \epsilon \quad \text{in} \quad O\left(\left[\frac{H_\nu D^{1+\nu}}{\epsilon}\right]^{\frac{2}{1+3\nu}} + \frac{\sigma^2 D^2}{\epsilon^2}\right) \text{ orac. calls.}$$

Implicit Variance Reduction

Problem Formulation

Problem: $F^* = \min_{x \in \operatorname{dom}\psi} [F(x) = f(x) + \psi(x)]$.

Assumptions:

1. f is (δ_f, L_f)-approximately smooth with components $(\bar{f}, \bar{g})$.
2. Bounded domain: $\|x - y\| \leq D$, $\forall x, y \in \operatorname{dom}\psi$.
3. f can be accessed only via unbiased stochastic oracle $\hat{g}$ for $\bar{g}$: $\mathbb{E}_\xi[g(x, \xi)] = \bar{g}(x)$.

Goal: Express complexity bounds in terms of $\sigma_*^2 := \operatorname{Var}_{\hat{g}}(x^*)$ instead of σ^2.

New assumption on variance

The variance $\operatorname{Var}_{\hat{g}}(x, y) := \mathbb{E}_\xi[\|[g(x, \xi) - g(y, \xi)] - [\bar{g}(x) - \bar{g}(y)]\|_*^2]$ is approximately smooth w.r.t. f:

$$\operatorname{Var}_{\hat{g}}(x, y) \leq 2L_{\hat{g}}[\beta_{f, \bar{f}, \bar{g}}(x, y) + \delta_{\hat{g}}].$$

Approximate Smoothness of Variance

Condition: $\mathrm{Var}_{\hat{g}}(x, y) \le 2L_{\hat{g}}[\beta_{f,\bar{f},\bar{g}}(x, y) + \delta_{\hat{g}}]$, where
$\mathrm{Var}_{\hat{g}}(x, y) \coloneqq \mathbb{E}_\xi[\|[g(x, \xi) - g(y, \xi)] - [\bar{g}(x) - \bar{g}(y)]\|_*^2]$.

Note:

- $\mathrm{Var}_{\hat{g}}(x, y)$ is the usual variance of $g(x, \xi) - g(y, \xi)$.
- If $\hat{g}_b$ is the mini-batch version of $\hat{g}$ of size b, then $\mathrm{Var}_{\hat{g}_b}(x, y) = \frac{1}{b}\mathrm{Var}_{\hat{g}}(x, y)$, and hence $L_{\hat{g}_b} = \frac{1}{b} L_{\hat{g}}$, $\delta_{\hat{g}_b} = \delta_{\hat{g}}$.

Main example: $f(x) = \mathbb{E}_\xi[f_\xi(x)]$, where each f_ξ is convex and (δ_ξ, L_ξ)-approx. smooth with components $(\bar{f}_\xi, \bar{g}_\xi)$. Then, $g(x, \xi) = \bar{g}_\xi(x)$ satisfies the variance condition with $\bar{f}(x) = \mathbb{E}_\xi[\bar{f}_\xi(x)]$, $\bar{g}(x) = \mathbb{E}_\xi[\bar{g}_\xi(x)]$, and $L_{\hat{g}} = L_{\max}$, $\delta_{\hat{g}} = \frac{1}{L_{\max}} \mathbb{E}_\xi[L_\xi \delta_\xi]$ $(\le \mathbb{E}_\xi[\delta_\xi])$, where $L_{\max} \coloneqq \sup_\xi L_\xi$.

Explanation:

$$\begin{aligned} \mathrm{Var}_{\hat{g}}(x, y) &\le \mathbb{E}_\xi[\|\bar{g}_\xi(x) - \bar{g}_\xi(y)\|_*^2] \le \mathbb{E}_\xi\left[2L_\xi\left(\beta_{f_\xi,\bar{f}_\xi,\bar{g}_\xi}(x, y) + \delta_\xi\right)\right] \\ &\le 2L_{\max}\left(\mathbb{E}_\xi[\beta_{f_\xi,\bar{f}_\xi,\bar{g}_\xi}(x, y)] + \delta_{\hat{g}}\right) = 2L_{\max}[\beta_{f,\bar{f},\bar{g}}(x, y) + \delta_{\hat{g}}]. \end{aligned}$$

Efficiency Bounds

NB: Consider the same methods as before (no modifications).

UniSgd: $$O\left(\frac{(L_f + L_{\hat{g}})D^2}{k} + \frac{\sigma_* D}{\sqrt{k}} + \delta_f + \delta_{\hat{g}}\right).$$

- When $\delta_f = \delta_{\hat{g}} = 0$, we recover the well-known rates for SGD with predefined stepsizes based on the knowledge of all the constants.

UniFastSgd: $$O\left(\frac{L_f D^2}{k^2} + \frac{L_{\hat{g}} D^2}{k} + \frac{\sigma_* D}{\sqrt{k}} + k\delta_f + \delta_{\hat{g}}\right).$$

- Different rates for L_f and $L_{\hat{g}}$ terms are unavoidable [Woodworth and Srebro 2021].
- For the special case $\delta_f = \delta_{\hat{g}} = 0$, similar results were obtained in [Woodworth and Srebro 2021; Ilandarideva et al. 2023] assuming that all constants are known.

Example: Problem with Hölder Smooth Components

Problem: $f(x) = \mathbb{E}_\xi[f_\xi(x)]$ with convex and $(\nu, H_\xi(\nu))$-Hölder smooth f_ξ.

Standard mini-batch oracle: $g_b(x, \xi_{[b]}) = \frac{1}{b} \sum_{j=1}^{b} \nabla f_{\xi_j}(x)$.

Method	Stochastic-Oracle (SO) Complexity
UniSgd	$\left(\frac{H_f(\nu) D^{1+\nu}}{\epsilon}\right)^{\frac{2}{1+\nu}} + \frac{1}{b} \min\left\{ \frac{\sigma^2 D^2}{\epsilon^2}, \left(\frac{H_{\max}(\nu)}{\epsilon}\right)^{\frac{2}{1+\nu}} D^2 + \frac{\sigma_*^2 D^2}{\epsilon^2} \right\}$
UniFastSgd	$\left(\frac{H_f(\nu) D^{1+\nu}}{\epsilon}\right)^{\frac{2}{1+3\nu}} + \frac{1}{b} \min\left\{ \frac{\sigma^2 D^2}{\epsilon^2}, \left(\frac{H_{\max}(\nu)}{\epsilon}\right)^{\frac{2}{1+\nu}} D^2 + \frac{\sigma_*^2 D^2}{\epsilon^2} \right\}$

Notation: $\sigma^2 = \sup\limits_{x \in \operatorname{dom} \psi} \operatorname{Var}_{\hat{g}_1}(x) \equiv \sup\limits_{x \in \operatorname{dom} \psi} \mathbb{E}_\xi[\|\nabla f_\xi(x) - \nabla f(x)\|_*^2]$, $\sigma_*^2 = \operatorname{Var}_{\hat{g}_1}(x^*) \equiv \mathbb{E}_\xi[\|\nabla f_\xi(x^*) - \nabla f(x^*)\|_*^2]$, $H_f(\nu)$ is the Hölder constant of degree ν for f.

Explicit Variance Reduction with SVRG

Universal SVRG

SVRG Oracle: $G(x,\xi) = g(x,\xi) - g(\tilde{x},\xi) + \bar{g}(\tilde{x})$.

Algorithm 3 $\mathrm{UniSvrg}_{\hat{g},\bar{g},\psi}(x_0; D)$

$\tilde{x}_0 = x_0$, $M_0 = 0$.
for $t = 0, 1, \ldots$ **do**
  $\hat{G}_t = \mathrm{SvrgOrac}_{\hat{g},\bar{g}}(\tilde{x}_t)$.
  $(\tilde{x}_{t+1}, x_{t+1}, M_{t+1}) \cong \mathrm{UniSgd}_{\hat{G}_t,\psi}(x_t, M_t, 2^{t+1}; D)$.

Algorithm 4 $\mathrm{UniSgd}_{\hat{g},\psi}(x_0, M_0, N; D)$

$g_0 \cong \hat{g}(x_0)$.
for $k = 0, \ldots, N-1$ **do**
  $x_{k+1} = \mathrm{Prox}_{\psi}(x_k, g_k, M_k)$, $g_{k+1} \cong \hat{g}(x_{k+1})$.
  $M_{k+1} = \sqrt{M_k^2 + \frac{1}{D^2}\|g_{k+1} - g_k\|_*^2}$.
return $(\bar{x}_N, x_N, M_N)$, where $\bar{x}_N := \frac{1}{N}\sum_{i=1}^N x_i$.

Universal Fast SVRG

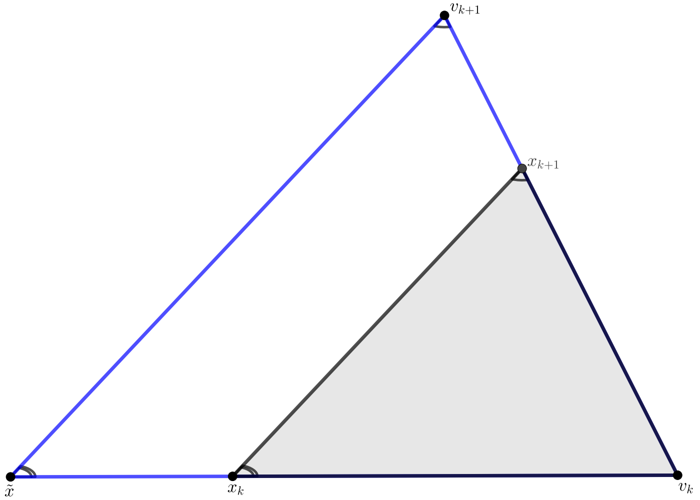
Algorithm 5 $\mathrm{UniFastSvrg}_{\hat{g},\bar{g},\psi}(x_0, N; D)$

$\tilde{x}_0 = \mathrm{argmin}_x\{\langle \bar{g}(x_0), x\rangle + \psi(x)\}$, $v_0 = x_0$, $M_0 = 0$, $A_0 = \frac{1}{N}$.
for $t = 0, 1, \ldots$ **do**
  $a_{t+1} = \sqrt{A_t}$, $A_{t+1} = A_t + a_{t+1}$.
  $(\tilde{x}_{t+1}, v_{t+1}, M_{t+1}) \cong \mathrm{UniTriSvrgEpoch}_{\hat{g},\bar{g},\psi}(\tilde{x}_t, v_t, M_t, A_t, a_{t+1}, N; D)$.

Algorithm 6 $\mathrm{UniTriSvrgEpoch}_{\hat{g},\bar{g},\psi}(\tilde{x}, v_0, M_0, A, a, N; D)$

$A_+ = A + a$, $x_0 = \frac{A}{A_+}\tilde{x} + \frac{a}{A_+}v_0$, $\hat{G} = \mathrm{SvrgOrac}_{\hat{g},\bar{g}}(\tilde{x})$, $G_{x_0} \cong \hat{G}(x_0)$.
for $k = 0, \ldots, N-1$ **do**
  $v_{k+1} = \mathrm{Prox}_\psi(v_k, G_{x_k}, M_k/a)$.
  $x_{k+1} = \frac{A}{A_+}\tilde{x} + \frac{a}{A_+}v_{k+1}$, $G_{x_{k+1}} \cong \hat{G}(x_{k+1})$.
  $M_{k+1} = \sqrt{M_k^2 + \frac{a^2}{D^2}\|G_{x_{k+1}} - G_{x_k}\|_*^2}$.
return $(\bar{x}_N, v_N, M_N)$, where $\bar{x}_N := \frac{1}{N}\sum_{k=1}^N x_k$.

Geometry of UniTriSvrgEpoch



Efficiency Guarantees

Method	Convergence rate	SO complexity
UniSgd	$\frac{L_f D^2}{k} + \delta_f + \min\left\{\frac{\sigma D}{\sqrt{k}}, \frac{\sigma_* D}{\sqrt{k}} + \frac{L_{\hat{g}} D^2}{k} + \delta_{\hat{g}}\right\}$	k
UniFastSgd	$\frac{L_f D^2}{k^2} + k\delta_f + \min\left\{\frac{\sigma D}{\sqrt{k}}, \frac{\sigma_* D}{\sqrt{k}} + \frac{L_{\hat{g}} D^2}{k} + \delta_{\hat{g}}\right\}$	k
UniSvrg	$\frac{(L_f + L_{\hat{g}}) D^2}{2^t} + \delta_f + \delta_{\hat{g}}$	$2^t + n \log t$
UniFastSvrg	$\frac{(L_f + L_{\hat{g}}) D^2}{n(t - \log\log n)^2} + t(\delta_f + \delta_{\hat{g}})$	nt

Note: Assuming that querying $\bar{g}$ is n times more expensive than $\hat{g}$.

Example: Problem with Hölder Smooth Components

Problem: $f(x) = \mathbb{E}_\xi[f_\xi(x)]$ with convex and $(\nu, H_\xi(\nu))$-Hölder smooth f_ξ.

Standard mini-batch oracle: $g_b(x, \xi_{[b]}) = \frac{1}{b}\sum_{j=1}^{b} \nabla f_{\xi_j}(x)$.

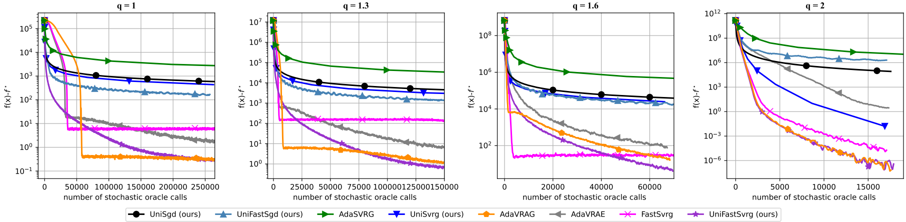
Method	Stochastic-Oracle (SO) Complexity
UniSgd	$\big(\frac{H_f(\nu)D^{1+\nu}}{\epsilon}\big)^{\frac{2}{1+\nu}} + \frac{1}{b}\min\big\{\frac{\sigma^2 D^2}{\epsilon^2}, \big(\frac{H_{\max}(\nu)}{\epsilon}\big)^{\frac{2}{1+\nu}} D^2 + \frac{\sigma_*^2 D^2}{\epsilon^2}\big\}$
UniFastSgd	$\big(\frac{H_f(\nu)D^{1+\nu}}{\epsilon}\big)^{\frac{2}{1+3\nu}} + \frac{1}{b}\min\big\{\frac{\sigma^2 D^2}{\epsilon^2}, \big(\frac{H_{\max}(\nu)}{\epsilon}\big)^{\frac{2}{1+\nu}} D^2 + \frac{\sigma_*^2 D^2}{\epsilon^2}\big\}$
UniSvrg	$\big[N_\nu(\epsilon) := \big(\frac{H_f(\nu)D^{1+\nu}}{\epsilon}\big)^{\frac{2}{1+\nu}} + \frac{1}{b}\big(\frac{H_{\max}(\nu)}{\epsilon}\big)^{\frac{2}{1+\nu}} D^2\big] + n_b \log_+ N_\nu(\epsilon)$
UniFastSvrg	$\big[\frac{n_b^\nu H_f(\nu)D^{1+\nu}}{\epsilon}\big]^{\frac{2}{1+3\nu}} + \big[\frac{n_b^\nu H_{\max}(\nu)D^{1+\nu}}{b^{(1+\nu)/2}\epsilon}\big]^{\frac{2}{1+3\nu}} + n_b \log\log n_b$

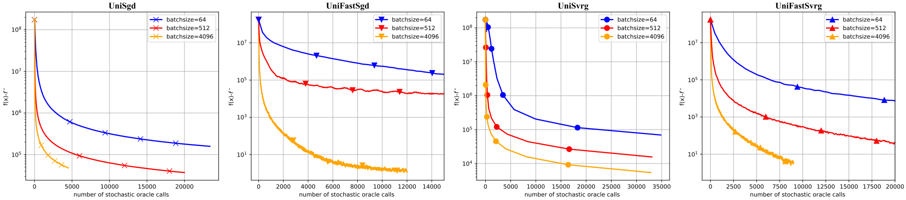
Note: Assuming that querying $\bar{g}$ is n_b times more expensive than $\hat{g}_b$.

Experiments & Conclusions

Experiments

Polyhderon feasibility problem: $\min\limits_{\|x\| \le R} \left\{ f(x) := \frac{1}{n} \sum\limits_{i=1}^{n} [\langle a_i, x \rangle - b_i]_+^q \right\}.$





Conclusions

- We showed that AdaGrad stepsizes can be applied, in a unified manner, in a large variety of situations, leading to universal methods suitable for multiple problem classes at the same time.
- The corresponding methods only need to know diameter D of feasible set, and automatically adapt to the best possible problem class described by various smoothness and variance assumptions.
- The universality is not for free: we need to know a good estimate of D. Adaptation to D could be addressed using the recently developed techniques from parameter-free methods. This is an important direction for future work.

Paper

Universality of AdaGrad Stepsizes for Stochastic Optimization: Inexact Oracle, Acceleration and Variance Reduction

arXiv:2406.06398

Thank you!

References I

- Y. Carmon and O. Hinder. Making SGD Parameter-Free. In **Proceedings of Thirty Fifth Conference on Learning Theory**, volume 178, pages 2360–2389, 2022.
- A. Cutkosky and K. A. Boahen. Online Learning Without Prior Information. In **Annual Conference Computational Learning Theory**, 2017.
- A. Cutkosky and F. Orabona. Black-Box Reductions for Parameter-free Online Learning in Banach Spaces. In **Annual Conference Computational Learning Theory**, 2018.
- A. Defazio and K. Mishchenko. Learning-Rate-Free Learning by D-Adaptation. In **Proceedings of the 40th International Conference on Machine Learning**, volume 202 of **Proceedings of Machine Learning Research**, pages 7449–7479, 2023.

References II

- O. Devolder, F. Glineur, and Y. Nesterov. First-order methods of smooth convex optimization with inexact oracle. **Mathematical Programming**, 146:37–75, 2013. DOI: 10.1007/s10107-013-0677-5.
- N. Doikov, K. Mishchenko, and Y. Nesterov. Super-universal regularized newton method. **SIAM Journal on Optimization**, 34(1):27–56, 2024.
- N. Doikov and Y. Nesterov. Minimizing uniformly convex functions by cubic regularization of newton method. **Journal of Optimization Theory and Applications**, 189(1):317–339, 2021.
- J. Duchi, E. Hazan, and Y. Singer. Adaptive subgradient methods for online learning and stochastic optimization. **Journal of Machine Learning Research**, 12:2121–2159, 2011.

References III

- A. Ene, H. L. Nguyen, and A. Vladu. Adaptive Gradient Methods for Constrained Convex Optimization and Variational Inequalities. In **Thirty-Fifth AAAI Conference on Artificial Intelligence**, pages 7314–7321, 2021.
- G. N. Grapiglia and Y. Nesterov. Regularized Newton methods for minimizing functions with Hölder continuous Hessians. **SIAM Journal on Optimization**, 27(1):478–506, 2017.
- G. N. Grapiglia and Y. Nesterov. Tensor Methods for Minimizing Convex Functions with Hölder Continuous Higher-Order Derivatives. **SIAM Journal on Optimization**, 30(4):2750–2779, 2020.
- S. Ilandarideva, A. Juditsky, G. Lan, and T. Li. Accelerated stochastic approximation with state-dependent noise. **arXiv preprint arXiv:2307.01497**, 2023.

References IV

A. Jacobsen and A. Cutkosky. Unconstrained online learning with unbounded losses. In **Proceedings of the 40th International Conference on Machine Learning**, 2023.

A. Kavis, K. Y. Levy, F. Bach, and V. Cevher. UniXGrad: A Universal, Adaptive Algorithm with Optimal Guarantees for Constrained Optimization. In **Advances in Neural Information Processing Systems 32**, pages 6260–6269. 2019.

K. Y. Levy, A. Yurtsever, and V. Cevher. Online Adaptive Methods, Universality and Acceleration. **Advances in Neural Information Processing Systems**, 31, 2018.

H. B. McMahan and M. Streeter. Adaptive Bound Optimization for Online Convex Optimization. **arXiv preprint arXiv:1002.4908**, 2010.

Y. Nesterov. Universal gradient methods for convex optimization problems. **Math. Program.**, 152:381–404, 2015.

References V

F. Orabona. Simultaneous model selection and optimization through parameter-free stochastic learning. In **Proceedings of the 27th International Conference on Neural Information Processing Systems**, pages 1116–1124, 2014.

A. Rodomanov, A. Kavis, Y. Wu, K. Antonakopoulos, and V. Cevher. Universal Gradient Methods for Stochastic Convex Optimization. **arXiv preprint arXiv:2402.03210**, 2024.

B. E. Woodworth and N. Srebro. An Even More Optimal Stochastic Optimization Algorithm: Minibatching and Interpolation Learning. **Advances in Neural Information Processing Systems**, 34:7333–7345, 2021.