Adaptive Gradient Methods for Stochastic Optimization

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Outline

Optimization Problems in Machine Learning

Optimization Problems

In Machine Learning (ML), we often deal with optimization problems:

$$f^* = \min_{x \in \mathbb{R}^d} f(x),$$

where $f\colon \mathbb{R}^d \to \mathbb{R}$ is a certain "objective function".

Main Example: Empirical Risk Minmization

- We are given a dataset $\mathcal{D} = \{(a_i, b_i)\}_{i=1}^n$ of n objects: a_i is the "features" of object i and b_i is a "label".
- Fix a certain model Φ which takes features a and has its own internal parameters (= "weights") $x \in \mathbb{R}^d$ and predicts a label $\hat{b} = \Phi(a; x)$.
- Define a "loss function" $L(\hat{b}, b)$ which measures how close is the prediction $\hat{b}$ from the "true label" b.

Goal: Solve the following optimization problem:

$$\min_{x \in \mathbb{R}^d} \Big[f(x) = \frac{1}{n} \sum_{i=1}^n \underbrace{L(\Phi(a_i; x), b_i)}_{=: f(x; a_i, b_i)} \Big].$$

Here $f(x; a, b)$ is the "loss" / "risk" of the model at object (a, b).

Examples

- (Least Squares) Regression problem, $b \in \mathbb{R}$.
  - Model: $\Phi(a; x) = \langle \phi(a), x \rangle$, where ϕ is a predefined feature transform.
  - Loss function: $L(\hat{b}, b) = \frac{1}{2}(\hat{b} - b)^2$.
  - Loss at object:

$$f(x; a, b) = \tfrac{1}{2}(\langle \phi(a), x \rangle - b)^2.$$

- (Logistic regression) Classification into $k \geq 2$ classes, $b \equiv (b^{(1)}, \dots, b^{(k)})$, $b^{(j)}$ is the prob. that object belongs to class j.
  - Model: Parameters $x = (x^{(1)}, \dots, x^{(k)})$, $x^{(j)} \in \mathbb{R}^{d_j}$, $x^{(k)} \equiv 0$, $\Phi(a; x) = \hat{b} \equiv (\hat{b}^{(1)}, \dots, \hat{b}^{(k)})$ with $\hat{b}^{(j)} = \frac{\exp(\langle \phi_j(a), x^{(j)} \rangle)}{\sum_{j'=1}^k \exp(\langle \phi_{j'}(a), x^{(j')} \rangle)}$, where ϕ_j are predefined feature transforms.
  - Loss function: Cross-entropy $L(\hat{b}, b) = -\sum_{j=1}^k b^{(j)} \ln \hat{b}^{(j)}$.
  - Loss at object:

$$f(x; a, b) = \ln\Bigl(\sum_{j=1}^k \exp(\langle \phi_j(a), x^{(j)} \rangle)\Bigr) - \sum_{j=1}^k b^{(j)} \langle \phi_j(a), x^{(j)} \rangle.$$

- (Deep Neural Networks, DNNs) Generalization of previous examples learning feature transforms ϕ_j "on-the-fly".

Gradient Descent (GD)

Algorithm

Problem: $\min_{x \in \mathbb{R}^d} f(x)$.

GD algorithm

Iterate for $k = 0, \ldots, T-1$:

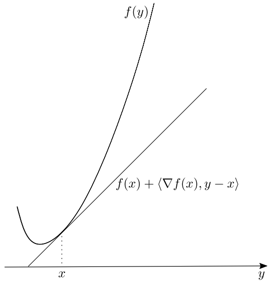
$$x_{k+1} = x_k - h \nabla f(x_k).$$

Here $\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_d}(x)\right) \in \mathbb{R}^d$ is the gradient of f at $x \in \mathbb{R}^d$, and $h > 0$ is the "stepsize" of the method.

How to choose h? How fast does this method converge?

Convex Functions

From now on, we assume the objective function f in our problem is convex:

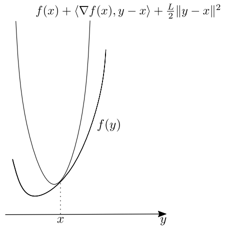


- Many basic ML models such as linear/logistic regression, SVM, etc. are convex.
- More advanced models such as DNNs are nonconvex, and we can only guarantee the convergence to a local minimizer (which is often sufficient in practice).
- Nonconvex function behaves like a convex one around a local minimizer, or when part of the variables are fixed.

Smooth Functions

Function f is called *L-smooth* ($L > 0$) if its gradient is L-Lipschitz:

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|, \qquad \forall x, y \in \mathbb{R}^d.$$


$f(x) + \langle \nabla f(x), y - x\rangle + \frac{L}{2}\|y - x\|^2$
$f(y)$
x
y

Equiv.: $f(y) \leq f(x) + \langle \nabla f(x), y - x\rangle + \frac{L}{2}\|y - x\|^2$.

Equivalent definition: $\|\nabla^2 f(x)\| \leq L$, $\forall x \in \mathbb{R}^d$, where $\nabla^2 f(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right)_{i,j=1}^d$ is the Hessian matrix.

Examples:

- (Quadratic function) $f(x) = \frac{1}{2}\langle Ax, x\rangle + \langle b, x\rangle$, $A \in \mathbb{S}^d_{++}$, $b \in \mathbb{R}^d$ $\implies L = \lambda_{\max}(A)$.
- (Log-sum-exp) $f(x) = \log(\sum_{i=1}^m e^{\langle a_j, x\rangle}) \implies L = \|A\|^2$, where $A = [a_1, \ldots, a_m]$.

Convergence Rate

Assumption: f is L-smooth.

Theorem (Section 2.1.5 in Nesterov 2018)

Consider GD with stepsize $h = \frac{1}{L}$*. Then, for any* k, $f(x_{k+1}) \leq f(x_k)$ *and*

$$f(x_T) - f^* \leq \frac{LR^2}{T},$$

where $R = \|x_0 - x^*\|$, x^* *is a minimizer of* f*.*

Main Drawback: Expensive Computations

Recall that, in ML problems, we typically solve

$$\min_{x \in \mathbb{R}^d} \left[f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x) \right],$$

where n is the number of objects and $f_i(x)$ is the loss at object i. In this case, computing the exact gradient

$$\nabla f(x) = \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(x)$$

is very expensive when n is big.

Natural idea: Approximate $\nabla f(x)$ by computing the average over only a few (randomly selected) objects.

Stochastic Gradient Method (SGD)

Stochastic Gradient Oracle (SGO)

SGO: Procedure taking $x \in \mathbb{R}^d$ and returning a random vector $g(x, \xi) \in \mathbb{R}^d$, where ξ is a random variable, g is a deterministic function, such that $g(x, \xi)$ is an unbiased estimate of $\nabla f(x)$:

$$\mathbb{E}_\xi[g(x, \xi)] = \nabla f(x).$$

Main example: If $f(x) = \frac{1}{n} \sum_{i=1}^n f_i(x)$, then

$$g(x, \xi) = \nabla f_\xi(x), \qquad \xi \cong \mathrm{Unif}(1, \dots, n).$$

More generally, if $f(x) = \mathbb{E}_\xi[F(x, \xi)]$, then $g(x, \xi) = \nabla_x F(x, \xi)$.

Variance of Stochastic Gradient

Variance: $\sigma_g^2(x) \coloneqq \mathbb{E}_\xi[\|g(x,\xi) - \nabla f(x)\|^2]$.

Mini-batching: Mini-batched version of g is an SGO g_b defined by

$$g_b(x, \xi_{[b]}) = \frac{1}{b} \sum_{j=1}^{b} g(x, \xi_b),$$

where $\xi_{[b]} = (\xi_1, \ldots, \xi_b)$ consists of b independent copies of ξ.

Key property: $\sigma_{g_b}^2(x) = \frac{1}{b}\sigma_g^2(x)$.

Mini-batching is especially useful when g_b can be computed in parallel.

SGD Algorithm

Problem: $f^* = \min_{x \in \mathbb{R}^d} f(x)$, where f is given by an SGO g.

SGD

Iterate for $k = 0, \dots, T-1$:

$$g_k = g(x_k, \xi_k),$$
$$x_{k+1} = x_k - h g_k.$$

Here $\xi_0, \dots, \xi_{T-1}$ are independent copies of ξ, and $h > 0$ is the stepsize of the method.

Convergence on Smooth Functions

Assumptions: f is L-smooth and $\sigma_g^2(x) \leq \sigma^2 \ \forall x \in \mathbb{R}^d$.

Output point: Either $\bar{x}_T = \frac{1}{T}\sum_{k=0}^{T-1} x_k$, or $\bar{x}_T \cong \mathrm{Unif}(x_0, \ldots, x_{T-1})$.

Theorem (Section 4.1.2 in Lan 2020)

Consider SGD with stepsize $h = \frac{1}{L + \frac{\sigma}{R}\sqrt{T}}$, where $R = \|x_0 - x^*\|$. Then,

$$\mathbb{E}[f(\bar{x}_T)] - f^* \leq \frac{LR^2}{T} + \frac{\sigma R}{\sqrt{T}}.$$

Note:

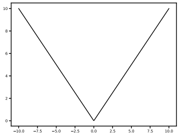
- First term is the rate of GD.
- Second term is due to stochastic noise and dominates when T is large enough.
- To accelerate convergence, we need to decrease σ (e.g., by mini-batching).

Nonsmooth Functions: Motivation

Many functions important in applications may be nonsmooth.
For example:

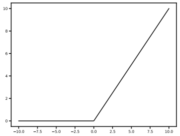
- Robust regression ($a_i \in \mathbb{R}^d$, $b \in \mathbb{R}$):

$$\min_{x \in \mathbb{R}^d} \left\{ f(x) = \frac{1}{n} \sum_{i=1}^{n} |\langle a_i, x \rangle - b_i| \right\}.$$



- SVM for binary classification ($a_i \in \mathbb{R}^d$, $b_i \in \{-1, 1\}$):

$$\min_{\|x\| \leq R} \left\{ f(x) = \frac{1}{n} \sum_{i=1}^{n} [1 - b_i \langle a_i, x \rangle]_+ \right\},$$



  where $R > 0$ and $[t]_+ := \max\{t, 0\}$ (also known as ReLU activation function for neural networks).

These functions are not smooth but still rather regular. They are examples of Lipschitz functions.

Lipschitz Functions

Function f is called M-Lipschitz if

$$|f(x) - f(y)| \leq M\|x - y\|, \qquad \forall x, y \in \mathbb{R}^d.$$

Equivalent condition[1]**:** $\|\nabla f(x)\| \leq M$, $\forall x \in \mathbb{R}^d$.

[1] $\nabla f(x)$ is an arbitrary subgradient of f at x if f is not differentible at this point.

Convergence Rate for Nonsmooth Functions

Assumptions: f is M-Lipschitz and $\sigma_g^2(x) \leq \sigma^2 \; \forall x \in \mathbb{R}^d$.

Theorem (Section 4.1.1 in Lan 2020)

Consider SGD with stepsize $h = \frac{R}{(M+\sigma)\sqrt{T}}$, *where* $R = \|x_0 - x^*\|$. *Then,*

$$\mathbb{E}[f(\bar{x}_T)] - f^* \leq \frac{MR}{\sqrt{T}} + \frac{\sigma R}{\sqrt{T}}.$$

Reminder: For L-smooth functions, we needed to choose $h = \frac{1}{L + \frac{\sigma}{R}\sqrt{T}}$ and the rate was $O(\frac{LR^2}{T} + \frac{\sigma R}{\sqrt{T}})$.

Intermediate Smoothness: Hölder Class

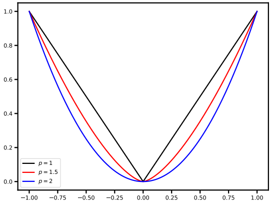
Lipschitz-smooth and nonsmooth Lipschitz functions are particular subclasses of the more general class of Hölder-smooth functions.

Function f is called (ν, H_ν)-Hölder smooth ($\nu \in [0,1]$ and $H_\nu > 0$) if

$$\|\nabla f(x) - \nabla f(y)\| \le H_\nu \|x - y\|^\nu, \qquad \forall x, y \in \mathbb{R}^d.$$

- Lipschitz-smooth functions ($\nu = 1$): $\|\nabla f(x) - \nabla f(y)\| \le H_1 \|x - y\|$.
- Lipschitz functions ($\nu = 0$): $\|\nabla f(x) - \nabla f(y)\| \le H_0$ ($H_0 = 2M$)

Example: $f(x) = \sum_{i=1}^{n} |\langle a_i, x\rangle - b_i|^p$ ($p \in [1,2]$) $\implies$ $\nu = p - 1$.



Convergence of SGD on Hölder-Smooth Functions

Assumptions: f is (ν, H_ν)-Hölder smooth and $\sigma_g^2(x) \leq \sigma^2 \ \forall x \in \mathbb{R}^d$.

Theorem

Consider SGD with stepsize $h \sim \frac{1}{\frac{H_\nu}{R^{1-\nu}} T^{\frac{1-\nu}{2}} + \frac{\sigma}{R}\sqrt{T}}$, *where* $R = \|x_0 - x^*\|$.

Then,

$$\mathbb{E}[f(\bar{x}_T)] - f^* \lesssim \frac{H_\nu R^{1+\nu}}{T^{\frac{1+\nu}{2}}} + \frac{\sigma R}{\sqrt{T}}.$$

Overparameterized Models

- Modern ML models (especially DNNs) are often overparameterized[2]: their number of parameters exceed the amount of training data, and the model can achieve (nearly) zero training loss.
- For such models, SGD works especially well, and convergence becomes comparable to GD while the cost of iteration is still significantly smaller.

[2] Cotter et al. 2011; Schmidt and Roux 2013; Needell et al. 2014; Ma et al. 2018; Liu and Belkin 2018; Necoara et al. 2019

Variance at Minimizer

To quantify how well the model fits the training data we can use the variance at the minimizer:

$$\sigma_*^2 := \sigma_g^2(x^*) \equiv \mathbb{E}_\xi[\|g(x,\xi)\|^2].$$

Example: Let $f(x) = \frac{1}{n}\sum_{i=1}^n f_i(x)$ and consider the "standard SGO"[3] $g(x,\xi) = \nabla f_\xi(x)$. Then,

$$\sigma_*^2 = \frac{1}{n}\sum_{i=1}^n \|\nabla f_i(x^*)\|^2.$$

If there exists a solution x^* such that it minimizes each loss function f_i, we have $\nabla f_i(x^*) = 0$ and $\sigma_* = 0$.

Main property: If each f_i is $L_{\max}$-smooth, then

$$\sigma_g^2(x) \leq 4L_{\max}[f(x) - f^*] + 2\sigma_*^2.$$

The first term in the above expression goes to zero as $f(x) \to f^*$.

[3] We could also use mini-batching but we do not do it for simplicity.

Convergence on Smooth Overparameterized Models

Assumptions: $f(x) \equiv \frac{1}{n}\sum_{i=1}^n f_i(x)$ with $L_{\max}$-smooth components f_i, standard SGO g.

Theorem

Consider SGD with stepsize $h \sim \frac{1}{L_{\max} + \frac{\sigma_*}{R}\sqrt{T}}$, *where* $R = \|x_0 - x^*\|$. *Then,*

$$\mathbb{E}[f(\bar{x}_T)] - f^* \lesssim \frac{L_{\max} R^2}{T} + \frac{\sigma_* R}{\sqrt{T}}.$$

Discussion:

- Previously, we had a similar result but with σ instead of σ^* and L instead of $L_{\max}$.
- If $\sigma_* = 0$, we can use a nearly constant step size $h \sim \frac{1}{L_{\max}}$ and get the $\frac{L_{\max}R^2}{T}$ convergence, which is similar to GD but each iteration is much cheaper.

Convergence on Hölder-Smooth Overparameterized Models

Assumptions: $f(x) \equiv \frac{1}{n}\sum_{i=1}^{n} f_i(x)$ with $(\nu, H_{\max}(\nu))$-Hölder smooth components, standard SGO g.

Theorem

Consider SGD with stepsize $h \sim \frac{1}{\frac{H_{\max}(\nu)}{R^{1-\nu}} T^{\frac{1-\nu}{2}} + \frac{\sigma_*}{R}\sqrt{T}}$, *where* $R = \|x_0 - x^*\|$.

Then,

$$\mathbb{E}[f(\bar{x}_T)] - f^* \lesssim \frac{H_{\max}(\nu) R^{1+\nu}}{T^{\frac{1+\nu}{2}}} + \frac{\sigma_* R}{\sqrt{T}}.$$

Summary

Case	Stepsize	Rate
M-Lipschitz, σ-variance	$\frac{M+\sigma}{R\sqrt{T}}$	$\frac{MR}{T} + \frac{\sigma R}{\sqrt{T}}$
L-smooth, σ-variance	$\frac{1}{L+\frac{\sigma}{R}\sqrt{T}}$	$\frac{LR^2}{T} + \frac{\sigma R}{\sqrt{T}}$
(ν, H_ν)-Hölder, σ-variance	$\frac{1}{\frac{H_\nu}{R^{1-\nu}} T^{\frac{1-\nu}{2}} + \frac{\sigma}{R}\sqrt{T}}$	$\frac{H_\nu R^{1+\nu}}{T^{\frac{1+\nu}{2}}} + \frac{\sigma R}{\sqrt{T}}$
$(\nu, H_{\max}(\nu))$-Hölder components	$\frac{1}{\frac{H_{\max}(\nu)}{R^{1-\nu}} T^{\frac{1-\nu}{2}} + \frac{\sigma_*}{R}\sqrt{T}}$	$\frac{H_{\max}(\nu) R^{1+\nu}}{T^{\frac{1+\nu}{2}}} + \frac{\sigma_* R}{\sqrt{T}}$

We will see next that adaptive methods such as AdaGrad can achieve all of this automatically (almost without tuning stepsize).

Adaptive Methods: AdaGrad

Classical AdaGrad

AdaGrad algorithm [Duchi et al. 2011]

Set $S_{-1} = 0$ and iterate for $k = 0, \dots, T-1$:

$$\begin{aligned} g_k &= g(x_k, \xi_k), \\ S_k^2 &= S_{k-1}^2 + g_k^2, \\ x_{k+1} &= x_k - \gamma \frac{g_k}{S_k}. \end{aligned}$$

Here $\gamma > 0$ is a parameter. All operations on vectors are component-wise.

NB: $S_k^2 = \sum_{t=0}^{k} g_t^2$ is the summation of squared gradients.

Adam [Kingma and Ba 2015]

Heuristical improvement over AdaGrad that often works well in practice.

Set $m_{-1} = 0$, $S_{-1} = 0$ and iterate for $k = 0, \dots, T-1$:

$$\begin{aligned} g_k &= g(x_k, \xi_k), \\ m_k &= \beta_1 m_{k-1} + (1-\beta_1) g_k, \qquad & \hat{m}_k &= \frac{m_k}{1-\beta_1^{k+1}}, \\ S_k^2 &= \beta_2 S_{k-1}^2 + (1-\beta_2) g_k^2, \qquad & \hat{S}_k^2 &= \frac{S_k^2}{1-\beta_2^{k+1}}, \\ x_{k+1} &= x_k - \alpha \frac{\hat{m}_k}{\hat{S}_k}. \end{aligned}$$

NB: $m_k = (1-\beta_1) \sum_{t=0}^k \beta_1^{k-t} g_t$ and $S_k^2 = (1-\beta_2) \sum_{t=0}^k \beta_2^{k-t} g_t^2$.

c.f.: heavy-ball method $x_{k+1} = x_k - \alpha g_k + \beta(x_k - x_{k-1})$ which can be written as $x_{k+1} = x_k - \alpha \sum_{t=0}^k \beta^{k-t} g_t$.

AdaGrad: Scalar Version

In what follows, we concentrate on the scalar AdaGrad method.

Scalar AdaGrad algorithm (also known as AdaGrad-Norm)

Set $S_{-1} = 0$ and iterate for $k = 0, \ldots, T-1$:

$$\begin{aligned} g_k &= g(x_k, \xi_k), \\ S_k^2 &= S_{k-1}^2 + \|g_k\|^2, \\ x_{k+1} &= x_k - \frac{\gamma}{S_k} g_k, \end{aligned}$$

- This is a simplification but sufficient to illustrate main points.
- Diagonal version is a natural “per-coordinate” extension of this idea. It approximates the gradient method $x_{k+1} = x_k - B^{-1} g_k$ with the fixed diagonal matrix B. This can be good in situations such as $[\nabla^2 f(x)]_{ii} \leq L_j$ with different L_j. E.g., if $f(x) = \frac{1}{2} \sum_{j=1}^{d} (a_j x^{(j)} - b_j)^2$, then a good scaling is $B_{jj} = L_j = a_j$.

AdaGrad with Projection

We introduce one more “minor modification” and consider from now on the following “safeguarded” version of AdaGrad:

$$\boxed{x_{k+1} = \pi_{B(x_0,R)}\Big(x_k - \frac{\gamma}{S_k} g_k\Big), \quad S_k^2 = \sum_{t=0}^{k} \|g_t\|^2,}$$

where $\pi_{B_R}(\cdot)$ is the projection onto the ball $B(x_0, R)$:

$$\pi_{B_R}(x) = \begin{cases} x, & \text{if } \|x - x_0\| \leq R, \\ x_0 + R\frac{x-x_0}{\|x-x_0\|}, & \text{otherwise,} \end{cases}$$

where $R \sim \|x_0 - x^*\|$.

Output point: Either $\bar{x}_T = \frac{1}{T}\sum_{k=0}^{T-1} x_k$, or $\bar{x}_T \cong \mathrm{Unif}(x_0, \dots, x_{T-1})$ (same as for SGD).

Convergence on Smooth and Nonsmooth Functions

Assumption: $\sigma_g^2(x) \leq \sigma^2 \ \forall x \in \mathbb{R}^d$.

Theorem (Levy et al. 2018)

Consider AdaGrad with $\gamma = R$, *where* $R \sim \|x_0 - x^*\|$.
If f *is* M*-Lipschitz, then*

$$\mathbb{E}[f(\bar{x}_T)] - f^* \lesssim \frac{MR}{\sqrt{T}} + \frac{\sigma R}{\sqrt{T}}.$$

If f *is* L*-smooth, then*

$$\mathbb{E}[f(\bar{x}_T)] - f^* \lesssim \frac{LR^2}{T} + \frac{\sigma R}{\sqrt{T}}.$$

NB: With the same parameter $\gamma = R$, AdaGrad works both for smooth and nonsmooth functions! And we don't even need to know M, L or σ (as in SGD).

Convergence on Hölder-Smooth Problems

Assumptions: f is (ν, H_ν)-Hölder smooth and $\sigma_g^2(x) \leq \sigma^2 \ \forall x \in \mathbb{R}^d$.

Theorem (Rodomanov et al. 2024)

Consider AdaGrad with $\gamma = R$, where $R \sim \|x_0 - x^*\|$. Then,

$$\mathbb{E}[f(\bar{x}_T)] - f^* \lesssim \frac{H_\nu R^{1+\nu}}{T^{\frac{1+\nu}{2}}} + \frac{\sigma R}{\sqrt{T}}.$$

NB: This is exactly the same convergence rate as we had for SGD with the carefully chosen stepsize (depending on ν, H_ν, R and σ).

Overparameterized Hölder-Smooth Problems

Assumptions: $f(x) \equiv \frac{1}{n} \sum_{i=1}^n f_i(x)$, where each f_i is $(\nu, H_{\max}(\nu))$-Hölder smooth, standard SGO g.

Theorem (Rodomanov et al. 2024)

Consider AdaGrad with $\gamma = R$, where $R \sim \|x_0 - x^*\|$. Then,

$$\mathbb{E}[f(\bar{x}_T)] - f^* \lesssim \frac{H_{\max}(\nu) R^{1+\nu}}{T^{\frac{1+\nu}{2}}} + \frac{\sigma_* R}{\sqrt{T}},$$

where $\sigma_* := \sigma_g(x^*)$.

NB: This is again the same convergence rate as for SGD, without any knowledge of ν, $H_{\max}(\nu)$ or σ_*.

Summary: Comparison with SGD

Case	Stepsize for SGD	γ in AdaGrad	Rate
M-Lipschitz, σ-variance	$\frac{M+\sigma}{R\sqrt{T}}$	R	$\frac{MR}{T} + \frac{\sigma R}{\sqrt{T}}$
L-smooth, σ-variance	$\frac{1}{L+\frac{\sigma}{R}\sqrt{T}}$	R	$\frac{LR^2}{T} + \frac{\sigma R}{\sqrt{T}}$
(ν, H_ν)-Hölder, σ-variance	$\frac{1}{\frac{H_\nu}{R^{1-\nu}} T^{\frac{1-\nu}{2}} + \frac{\sigma}{R}\sqrt{T}}$	R	$\frac{H_\nu R^{1+\nu}}{T^{\frac{1+\nu}{2}}} + \frac{\sigma R}{\sqrt{T}}$
$(\nu, H_{\max}(\nu))$-Hölder components	$\frac{1}{\frac{H_{\max}(\nu)}{R^{1-\nu}} T^{\frac{1-\nu}{2}} + \frac{\sigma_*}{R}\sqrt{T}}$	R	$\frac{H_{\max}(\nu) R^{1+\nu}}{T^{\frac{1+\nu}{2}}} + \frac{\sigma_* R}{\sqrt{T}}$

Conclusions

Conclusions

- Stepsize and convergence rate of SGD depend on many characteristic of the specific problem: smoothness, variance, degree of overparameterization, . . .
- Adaptive methods such as AdaGrad reduce the knowledge of parameters to one and are “universal”—they automatically adapt to the best possible setting for a specific problem.
- Our theory provides a possible explanation why adaptive methods often perform well in practice.

Thank you!

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